

# NON-PROPERLY EMBEDDED MINIMAL PLANES IN HYPERBOLIC 3-SPACE

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**ABSTRACT.** In this paper, we show that there are non-properly embedded minimal surfaces with finite topology in a simply connected Riemannian 3-manifold with nonpositive curvature. We show this result by constructing a non-properly embedded minimal plane in  $\mathbf{H}^3$ . Hence, this gives a counterexample to Calabi-Yau conjecture for embedded minimal surfaces in negative curvature case.

## 1. INTRODUCTION

In the last decade, there have been a great progress in the understanding of the global structure of complete, embedded minimal surfaces in  $\mathbf{R}^3$ . In [CM], Colding and Minicozzi showed that a complete, embedded minimal surfaces with finite topology in  $\mathbf{R}^3$  is proper. This result is also known as Calabi-Yau conjecture for embedded minimal surfaces in the literature. Later, Meeks and Rosenberg generalized this result to complete, embedded minimal surfaces with positive injectivity radius in  $\mathbf{R}^3$  in [MR1]. A nice survey on Calabi-Yau problem can be found at [Al].

If we consider the question in more general settings, it is not hard to construct examples of non-properly embedded minimal surfaces with finite topology in a non-simply connected 3-manifold, or in a simply connected 3-manifold with some arbitrary metric. However, there is no known example of a non-properly embedded minimal surface with finite topology in a simply connected 3-manifold with nonpositive curvature (See Final Remarks).

On the other hand, the key lemma of [MR1] to prove its main result also applies to all 3-manifolds with nonpositive curvature. Hence, the question of whether the generalization of Calabi-Yau conjecture for embedded minimal surfaces to simply connected 3-manifolds with nonpositive curvature is true or not became interesting (See Final Remarks). In this paper, we construct an example of non-properly embedded minimal plane in  $\mathbf{H}^3$  which shows that the Calabi-Yau conjecture for embedded minimal surfaces does not generalize to simply connected 3-manifolds with nonpositive curvature.

**Theorem 2.1** There exists a non-properly embedded, complete minimal plane in  $\mathbf{H}^3$ .

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This example is inspired by the heuristic construction of Freedman and He mentioned also in [Ga]. Note that Meeks and Perez also mentioned such an example in their survey paper [MP]. The idea is as follows: Take a sequence of round circles  $\{C_n\}$  in  $S_\infty^2(\mathbf{H}^3)$  which limits on the equator circle. Each circle  $C_n$  bounds a geodesic plane  $P_n$ . By connecting each circle  $C_n$  with  $C_{n+1}$  by using bridges in  $S_\infty^2(\mathbf{H}^3)$ , we get a nonrectifiable curve  $\Gamma$  in  $S_\infty^2(\mathbf{H}^3)$  (See Figure 1). Then, we construct a special sequence of minimal disks  $\{E_n\}$  with  $\partial E_n \rightarrow \Gamma$  and show that the limiting minimal plane  $\Sigma$  with  $\partial_\infty \Sigma = \Gamma$  does not stay close to  $S_\infty^2(\mathbf{H}^3)$  by using barrier tunnels. Then, we prove that  $\Sigma$  is a non-properly embedded minimal plane in  $\mathbf{H}^3$ . Intuitively, one can imagine  $\Sigma$  as the collection of geodesic planes  $\{P_n\}$  which are connected via bridges at infinity (See Figure 3).

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## 2. THE CONSTRUCTION

First, we need a few definitions. Basic notions and results which will be used in this paper can be found in the survey article [Co2].

**Definition 2.1.** A *least area disk* in a Riemannian manifold  $M$  is a compact disk which has the smallest area among the disks in  $M$  with the same boundary. A *least area plane* is a complete plane such that any compact subdisk in the plane is a least area disk. A *least area annulus* (compact case) is the annulus which has the least area among the annuli with the same boundary. A *minimal surface* is a surface whose mean curvature vanishes everywhere.

We will finish the construction in 3 steps. First, we construct a special sequence of compact minimal disks  $\{E_n\}$  which will limit on a minimal plane. In second part, we show that the limit of  $\{E_n\}$  give us a complete, embedded minimal plane  $\Sigma$  in  $\mathbf{H}^3$ . Finally, we show that  $\Sigma$  is indeed non-properly embedded in  $\mathbf{H}^3$ .

### 2.1. The Sequence.

In this step, we will construct a sequence of minimal disks  $\{E_n\}$  in  $\mathbf{H}^3$  which will give us a complete embedded minimal plane in  $\mathbf{H}^3$  as the limit in the next step.

First, consider the half space model for  $\mathbf{H}^3$ . Here  $S_\infty^2(\mathbf{H}^3) = \mathbf{R}^2 \times \{0\} \cup \{\infty\}$ . Now, define a sequence of round circles  $\{C_n\}$  in  $S_\infty^2(\mathbf{H}^3)$  such that  $C_n$  is the round circle in  $\mathbf{R}^2 \times \{0\}$  with center at origin and radius  $r_n = 1 + \frac{1}{n}$ . Hence, the sequence limits on the unit circle in  $\mathbf{R}^2 \times \{0\}$ .

Now, we want to connect each consecutive circle  $C_n$  and  $C_{n+1}$  with thin bridges in  $S_\infty^2(\mathbf{H}^3)$ . Let's start with  $C_1$  and  $C_2$ . Like in [Ha], we construct a tunnel in  $\mathbf{H}^3$  as follows. Let  $\eta_1^+$  and  $\eta_1^-$  be sufficiently small round circles in  $\mathbf{R}^2 \times \{0\}$  with radius  $\delta_1$  where  $2\delta_1 < r_1 - r_2$ . Let the center of  $\eta_1^+$  be  $(\frac{r_1+r_2}{2}, \epsilon_1)$  and let the center of  $\eta_1^-$  be  $(\frac{r_1+r_2}{2}, -\epsilon_1)$  in  $\mathbf{R}^2$  where  $\epsilon_1 > \delta_1 > 0$ . Let  $P_1^\pm$  be the geodesic plane in  $\mathbf{H}^3$  with  $\partial_\infty P_1^\pm = \eta_1^\pm$ . By changing  $\epsilon_1$  and  $\delta_1$  if necessary, we can find sufficiently large disk  $D_1^\pm$  in  $P_1^\pm$  with  $\partial D_1^\pm = \beta_1^\pm$  such that  $\beta_1^+ \cup \beta_1^-$

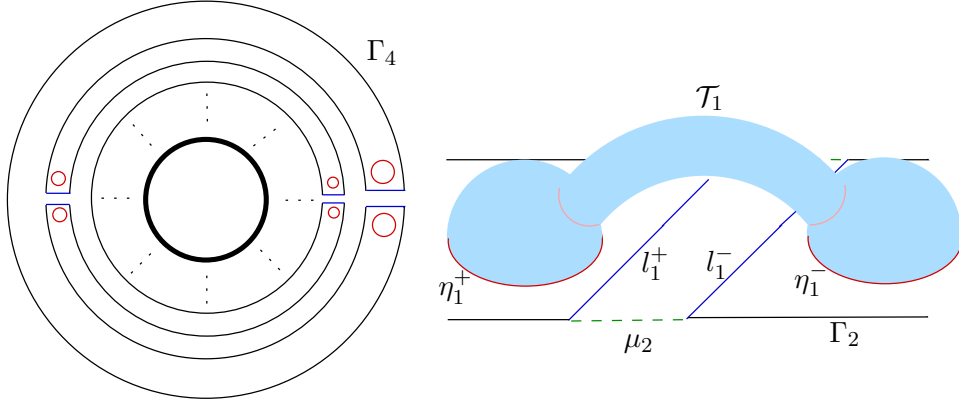


FIGURE 1. In the figure left,  $\Gamma_4$  is constructed by using the circles  $C_1, C_2, C_3, C_4$  in  $S_\infty^2(\mathbf{H}^3)$  by connecting each other with bridges (blue line segments  $l_i^\pm$ ). The red circles represents  $\eta_i^\pm$ . In the figure right, the tunnel  $\mathcal{T}_1$  is shown.

bounds a least area annulus  $A_1$  in  $\mathbf{H}^3$  like in [Ha]. Define the annulus  $\mathcal{A}_1$  such that  $\mathcal{A}_1 = (P_1^+ - D_1^+) \cup (P_1^- - D_1^-) \cup A_1$ . Then,  $\mathcal{A}_1$  separates  $\mathbf{H}^3$  into two components. Let  $\mathcal{T}_1$  be the component with  $\partial_\infty \mathcal{T}_1 = \Delta_1^+ \cup \Delta_1^-$  where  $\Delta_1^\pm$  is the disk in  $\mathbf{R}^2$  with boundary  $\eta_1^\pm$ . We will call  $\mathcal{T}_1$  as a *tunnel*. Note that  $\partial \mathcal{T}_1 = A_1$  and  $\partial_\infty \mathcal{T}_1 = \Delta_1^+ \cup \Delta_1^-$ .

Now, we will connect  $C_1$  and  $C_2$  with a bridge. Let  $\mu_1$  be an arc segment in  $C_1$  with  $\mu_1 = C_1 \cap \left([1, 3] \times \left(-\frac{\epsilon_1 - \delta_1}{2}, \frac{\epsilon_1 - \delta_1}{2}\right)\right)$  and  $\mu_2$  be an arc segment in  $C_2$  with  $\mu_2 = C_2 \cap \left([1, 3] \times \left(-\frac{\epsilon_1 - \delta_1}{2}, \frac{\epsilon_1 - \delta_1}{2}\right)\right)$ . Let the end points of  $C_n - \mu_n$  be  $p_n^+$  and  $p_n^-$  where  $p_n^+$  be the endpoint belonging to the upper half space in  $\mathbf{R}^2$ . Let  $l_1^+$  be the straight line segment in  $\mathbf{R}^2$  between  $p_1^+$  and  $p_2^+$ . Likewise, let  $l_1^-$  be the straight line segment in  $\mathbf{R}^2$  between  $p_1^-$  and  $p_2^-$ . Now, define  $C_1 \# C_2$  such that  $C_1 \# C_2 = (C_1 - \mu_1) \cup (C_2 - \mu_2) \cup l_1^+ \cup l_1^-$ . Let  $\Gamma_1 = C_1$  and  $\Gamma_2 = C_1 \# C_2$ . Notice that  $\Gamma_2$  separates  $S_\infty^2(\mathbf{H}^3)$  into two parts, where one of the parts contains  $\Delta_1^+$  and  $\Delta_1^-$  which we call the base of the tunnel  $\mathcal{T}_1$ . In the Poincaré ball model for  $\mathbf{H}^3$ , one can think that the bridge constructed with  $l_1^+ \cup l_1^-$  goes over the tunnel  $\mathcal{T}_1$ .

Next, we will connect  $\Gamma_2$  and  $C_3$  with a bridge. This time, the bridge will be constructed in the opposite side, i.e. near the line segment  $[-r_3, -r_2] \times \{0\}$  line segment in negative  $x$ -axis of  $\mathbf{R}^2$ . Let  $\lambda_1 = \frac{r_2 - r_3}{r_1 - r_2}$ . Now, do the same construction described in the previous paragraphs for  $\epsilon_2 = \lambda_1 \cdot \epsilon_1$  and  $\delta_2 = \lambda_1 \cdot \delta_1$ . Here,  $\eta_2^\pm$  will be the round circles in  $\mathbf{R}^2$  with radius  $\delta_2$  and center  $(-\frac{r_2 + r_3}{2}, \pm \epsilon_2)$ . Let  $\phi_2$  be the isometry of  $\mathbf{H}^3$  with  $\phi_1(\eta_1^\pm) = \eta_2^\pm$  and  $\phi_1((\frac{r_1 + r_2}{2}, 0)) = (\frac{r_2 + r_3}{2}, 0)$ . Notice that  $\phi_2$  can be obtained by composing the parabolic isometry which translates the point  $(\frac{r_1 + r_2}{2}, 0)$  to the point  $(\frac{r_2 + r_3}{2}, 0)$  fixing  $\{\infty\}$ , and the hyperbolic isometry with dilation constant  $\lambda_1$ . Then, define the second tunnel  $\mathcal{T}_2$  as the isometric image of  $\mathcal{T}_1$ , i.e.  $\mathcal{T}_2 = \phi_2(\mathcal{T}_1)$ . Define annuli  $A_2$  and  $\mathcal{A}_2$  accordingly. The second

bridge can be defined similarly. Let  $\tau_2 = C_2 \cap \left( [-3, -1] \times \left( -\frac{\epsilon_2 - \delta_2}{2}, \frac{\epsilon_2 - \delta_2}{2} \right) \right)$  and  $\tau_3 = C_3 \cap \left( [-3, -1] \times \left( -\frac{\epsilon_2 - \delta_2}{2}, \frac{\epsilon_2 - \delta_2}{2} \right) \right)$ . For  $n \geq 2$ , let the end points of  $C_n - \tau_n$  be  $q_n^+$  and  $q_n^-$  where  $q_n^+$  be the endpoint belonging to the upper half space in  $\mathbf{R}^2$ . Let  $l_2^+$  be the straight line segment in  $\mathbf{R}^2$  between  $q_2^+$  and  $q_3^+$ . Similarly,  $l_2^-$  be the straight line segment in  $\mathbf{R}^2$  between  $q_2^-$  and  $q_3^-$ . Like before define  $\Gamma_2 \# C_3$  such that  $\Gamma_2 \# C_3 = (\Gamma_2 - \tau_2) \cup (C_3 - \tau_3) \cup l_2^+ \cup l_2^-$ . Let  $\Gamma_3 = \Gamma_2 \# C_3$ .

Hence, if we continue the process, we can define the tunnels and bridges for any  $n$  as follows. Let  $o_n$  be the largest odd number which is smaller than or equal to  $n$ . Let  $e_n$  be the largest even number which is smaller than or equal to  $n$ . In other words, if  $n$  is odd,  $o_n = n$ , and if  $n$  is even  $o_n = n - 1$ . Similarly, if  $n$  is even,  $e_n = n$  and if  $n$  is odd,  $e_n = n - 1$ .

Now, let's define the basic components in the construction of bridges and tunnels in general terms. Let  $\lambda_n = \frac{r_{n+1} - r_{n+2}}{r_1 - r_2}$ . Then define  $\epsilon_n = \lambda_n \cdot \epsilon_1$  and  $\delta_n = \lambda_n \cdot \delta_1$ .  $\mu_n$  is an arc segment in the circle  $C_n$  such that  $\mu_n = C_n \cap \left( [1, 3] \times \left( -\frac{\epsilon_{o_n} - \delta_{o_n}}{2}, \frac{\epsilon_{o_n} - \delta_{o_n}}{2} \right) \right)$ . Similarly,  $\tau_n$  is an arc segment in the circle  $C_n$  such that  $\tau_n = C_n \cap \left( [-3, -1] \times \left( -\frac{\epsilon_{e_n} - \delta_{e_n}}{2}, \frac{\epsilon_{e_n} - \delta_{e_n}}{2} \right) \right)$ . Then when  $n$  is odd,  $l_n^\pm$  would be the straight line segments between the points  $p_n^\pm$  and  $p_{n+1}^\pm$ , and when  $n$  is even,  $l_n^\pm$  would be the straight line segments between the points  $q_n^\pm$  and  $q_{n+1}^\pm$ . Hence, the arc segments  $\mu_n$ , and odd indexed line segments  $l_n^\pm$  live in the right side ( $x > 0$ ) of  $\mathbf{R}^2$ , and the arc segments  $\tau_n$  and even indexed line segments  $l_n^\pm$  live in the left side ( $x < 0$ ) of  $\mathbf{R}^2$ .

Define  $\Gamma_n$  inductively such that  $\Gamma_{n+1} = \Gamma_n \# C_{n+1}$ . Here  $\Gamma_n \# C_{n+1}$  can be defined as follows. When  $n$  is odd,  $\Gamma_n \# C_{n+1} = (\Gamma_n - \mu_n) \cup (C_{n+1} - \mu_{n+1}) \cup l_n^+ \cup l_n^-$ . When  $n$  is even,  $\Gamma_n \# C_{n+1} = (\Gamma_n - \tau_n) \cup (C_{n+1} - \tau_{n+1}) \cup l_n^+ \cup l_n^-$ . By iterating the procedure, we will get a non-rectifiable connected curve  $\Gamma$  (or  $\Gamma_\infty$ ) in  $S_\infty^2(\mathbf{H}^3)$  which has infinite length.

We define the tunnels as follows. When  $n$  is odd, the round circles  $\eta_n^\pm$  would be the round circles of radius  $\delta_n$  with center  $(\frac{r_n + r_{n+1}}{2}, \pm \epsilon_n)$ . Similarly, when  $n$  is even, the round circles  $\eta_n^\pm$  would be the round circles of radius  $\delta_n$  with center  $(-\frac{r_n + r_{n+1}}{2}, \pm \epsilon_n)$ . Then  $P_n^\pm$  would be the geodesic planes in  $\mathbf{H}^3$  with  $\partial_\infty P_n^\pm = \eta_n^\pm$ . Define the least area annulus  $A_n$  such that the boundary curves are in the geodesic planes  $P_n^+$  and  $P_n^-$ . Hence, the tunnel  $\mathcal{T}_n$  can be defined accordingly by using  $P_n^\pm$  and  $A_n$ . Notice that when  $n$  is odd, the tunnels  $\mathcal{T}_n$  (and hence  $\eta_n^\pm$ ,  $P_n^\pm$ ,  $A_n$ ) are in the right side ( $x > 0$ ) in  $\mathbf{R}^2$ , and when  $n$  is even, the tunnels  $\mathcal{T}_n$  (and hence  $\eta_n^\pm$ ,  $P_n^\pm$ ,  $A_n$ ) are in the left side ( $x < 0$ ) in  $\mathbf{R}^2$ . In other words, odd indexed tunnels and bridges are in the right side of  $\mathbf{R}^2$  and even indexed tunnels and bridges are in the left side of  $\mathbf{R}^2$ .

Now, we construct the sequence of embedded minimal disks  $\{E_n\}$  in  $\mathbf{H}^3$ . Let  $X_1 = \mathbf{H}^3 - \text{int}(\mathcal{T}_1)$ . Then  $\partial X_1 = \mathcal{A}_1$  and  $\partial_\infty X_1 = S_\infty^2(\mathbf{H}^3) - (\Delta_1^+ \cup \Delta_1^-)$ . In general, let  $X_n = \mathbf{H}^3 - \bigcup_{i=1}^n \text{int}(\mathcal{T}_i)$ . Then,  $\partial X_n = \bigcup_{i=1}^n \mathcal{A}_i$  and  $\partial_\infty X_n = S_\infty^2(\mathbf{H}^3) - \bigcup_{i=1}^n \text{int}(\Delta_i^+ \cup \Delta_i^-)$ . Define  $X$  such that  $X = \bigcap_{i=1}^\infty X_n = \mathbf{H}^3 - \bigcup_{i=1}^\infty \text{int}(\mathcal{T}_i)$ .

Let  $v$  be the circle of radius 3 with center at origin in  $\mathbf{R}^2$ . Let  $\mathcal{P}$  be the geodesic plane in  $\mathbf{H}^3$  with  $\partial_\infty \mathcal{P} = v$ . Then  $\mathcal{P}$  separates  $\mathbf{H}^3$  into two components. Let  $\Omega$  be the component whose asymptotic boundary contains the origin. Also, define the horoball  $\mathcal{H}_i$  such that  $\mathcal{H}_i = \{ (x, y, z) \in \mathbf{H}^3 \mid z \geq \frac{1}{i} \}$ . Let  $S_i$  be the horosphere with  $S_i = \partial \mathcal{H}_i$ , i.e.  $S_i = \{ (x, y, \frac{1}{i}) \in \mathbf{H}^3 \}$ . Then define the domain  $\Omega_i$  such that  $\Omega_i = \Omega \cap \mathcal{H}_i \cap X$ . Notice that  $\Omega_i$  is a compact mean convex domain since the horosphere  $S_i$  has mean curvature 1, and the geodesic plane  $\mathcal{P}$  and the least area annulus  $\mathcal{A}_n$  have mean curvature 0.

Let  $p$  be the point  $(0, 0, 1)$  in the upper half space model of  $\mathbf{H}^3$ . Notice that  $\{p\} = \Sigma_1 \cap l$  where  $\Sigma_1$  is the geodesic plane in  $\mathbf{H}^3$  with  $\partial_\infty \Sigma_1 = \Gamma_1$ , and  $l$  is the vertical geodesic  $l = \{ (0, 0, t) \in \mathbf{H}^3 \mid t > 0 \}$ . Let  $\mathcal{C}_n$  be the geodesic cone over  $\Gamma_n$  in  $S_\infty^2(\mathbf{H}^3)$  with cone point  $p$ . In other words, if  $\gamma_{pq}$  is the geodesic ray in  $\mathbf{H}^3$  starting from  $p$  and limiting on  $q \in S_\infty^2(\mathbf{H}^3)$ , then  $\mathcal{C}_n = \bigcup_{q \in \Gamma_n} \gamma_{pq}$ . Let  $\alpha_n^i$  be a simple closed curve in  $\mathbf{H}^3$  defined as the intersection of the horosphere  $S_i$  and the geodesic cone  $\mathcal{C}_n$ , i.e.  $\alpha_n^i = S_i \cap \mathcal{C}_n$ . Notice that for any  $n$ , there exists a number  $c_n > 0$  such that if  $i \geq c_n$  then  $\alpha_n^i \cap \mathcal{T}_n = \emptyset$ . Now, we need a lemma due to Meeks-Yau in order to continue to the construction.

**Lemma 2.1.** [MY] *Let  $\Omega$  be a compact, mean convex 3-manifold, and  $\alpha \subset \partial\Omega$  be a nullhomotopic simple closed curve. Then, there exists an embedded least area disk  $D \subset M$  with  $\partial D = \alpha$ .*

By construction, for any  $i$ ,  $\Omega_i$  is a compact, mean convex 3-manifold. Also, for any  $n$ ,  $\alpha_n^{c_n}$  is a simple closed curve in  $\partial\Omega_{c_n}$ . Hence, by Lemma 2.1, there exists an embedded least area disk  $E_n$  in  $\Omega_{c_n}$  such that  $\partial E_n = \alpha_n^{c_n}$  (say  $\alpha_n$  for short). Note that being least area in  $\Omega_{c_n}$  does not imply that  $E_n$  is also least area in  $\mathbf{H}^3$ . Even though  $E_n$  may not be a least area disk in  $\mathbf{H}^3$ , we claim that it is indeed least area in  $X = \mathbf{H}^3 - \bigcup_{i=1}^\infty \text{int}(\mathcal{T}_i)$ .

**Lemma 2.2.** *For any  $n$ ,  $E_n$  is a least area disk in  $X$ .*

*Proof:* We know that  $E_n$  is a least area disk in  $\Omega_{c_n}$  by construction. The only case where  $E_n$  is not a least area disk in  $X$  is the existence of another disk  $\mathcal{D}$  in  $X$  with  $\partial \mathcal{D} = \partial E_n = \alpha_n$  and  $|\mathcal{D}| < |E_n|$  where  $|\cdot|$  represents the area. Notice that the horoball  $\mathcal{H}_i$  and the half space  $\Omega$  are convex subsets of  $\mathbf{H}^3$ . Hence, for any  $i$ ,  $\hat{\Omega}_i = \mathcal{H}_i \cap \Omega$  is a convex subset of  $\mathbf{H}^3$ . Then if  $\pi_i : \mathbf{H}^3 \rightarrow \hat{\Omega}_i$  is the nearest point projection, since  $\hat{\Omega}_i$  is convex,  $\pi_i$  would be distance reducing. Let  $\hat{\mathcal{D}} = \pi_{c_n}(\mathcal{D})$  (say  $\pi$  for  $\pi_{c_n}$  for short). Then,  $|\hat{\mathcal{D}}| \leq |\mathcal{D}|$  and  $\hat{\mathcal{D}}$  is a disk in  $\hat{\Omega}_{c_n}$  with  $|\hat{\mathcal{D}}| \leq |\mathcal{D}| < |E_n|$ .

The outline of the proof is as follows: If we can show that  $\hat{\mathcal{D}}$  is disjoint from any tunnel  $\mathcal{T}_i$ , then we get a contradiction, as  $\hat{\mathcal{D}}$  would be a disk in  $\Omega_{c_n}$  with  $\partial \hat{\mathcal{D}} = \alpha_n$ , and  $|\hat{\mathcal{D}}| < |E_n|$ . Hence, if  $\hat{\mathcal{D}}$  intersects some tunnels  $\mathcal{T}_i$ , we will do a surgery on  $\hat{\mathcal{D}}$  by removing the subdisks where  $\hat{\mathcal{D}}$  intersect the tunnels, and get a new disk  $\hat{\mathcal{D}}'$  in  $\Omega_{c_n}$  with  $\partial \hat{\mathcal{D}}' = \alpha_n$  with  $|\hat{\mathcal{D}}'| < |E_n|$ . Since  $E_n$  is the least area disk in  $\Omega_{c_n}$  with boundary  $\alpha_n$ , this will give us a contradiction.

Notice that  $\pi(x) = x$  for any  $x \in \mathcal{D} \cap \text{int}(\Omega_{c_n})$ , and  $\pi(\mathcal{D} - \text{int}(\Omega_{c_n})) \subset S_{c_n}$ . Let  $T_i \cap S_{c_n} = O_i^+ \cup O_i^-$ . Hence, if we can show that  $\mathcal{D} \cap O_i^\pm = \emptyset$ , we are done.

First, let  $\psi : D^2 \rightarrow X$  be a parametrization of  $\mathcal{D}$ , i.e.  $\psi(D^2) = \mathcal{D}$ . Then let  $\varphi : D^2 \rightarrow \Omega_{c_n}$  be the parametrization of  $\widehat{\mathcal{D}}$  with  $\varphi = \pi \circ \psi$ , i.e.  $\varphi(D^2) = \widehat{\mathcal{D}}$ . If  $\widehat{\mathcal{D}} \cap O_i^+ \neq \emptyset$ , there are two cases:  $O_i^+ \not\subset \widehat{\mathcal{D}}$  or  $O_i^+ \subset \widehat{\mathcal{D}}$ . If  $\widehat{\mathcal{D}} \cap O_i^+ \neq \emptyset$  and  $O_i^+ \not\subset \widehat{\mathcal{D}}$ , then let  $V_i^+ = \varphi^{-1}(O_i^+ \cap \widehat{\mathcal{D}})$ . Since  $T_i$  is separating,  $\partial V_i^+$  would be a collection of circles. Since  $O_i^+ \not\subset \widehat{\mathcal{D}}$ ,  $\varphi(\partial V_i^+)$  is a collection of nonessential circles in  $\partial T_i$ . Hence, we can push off  $O_i^+ \cap \widehat{\mathcal{D}}$ , and get another disk  $\widehat{\mathcal{D}}'$  with less area.

Now, if  $O_i^+ \subset \mathcal{D}$ , then we claim that  $\varphi^{-1}(O_i^+)$  consists of even number of disks after removing the nonessential circles like in the previous paragraph. Let  $y$  be a point in  $O_i^+$ .  $\pi^{-1}(y)$  would be a geodesic ray  $\rho$  starting at  $y$  and orthogonal to  $S_i$ . Also let  $\tau$  be an infinite ray in  $T_i$  starting at  $y$  and outside of  $\mathcal{H}_{c_n}$ . We know that  $\mathcal{D} \cap \tau = \emptyset$  as  $\mathcal{D} \subset X$ .

Let  $\Delta$  be a homotopy between the infinite rays  $\tau$  and  $\rho$ . In other words,  $\Delta : [0, 1] \times [0, 1] \rightarrow \mathbf{H}^3$  a continuous map such that  $\Delta(\{0\} \times [0, 1]) = \tau$ ,  $\Delta(\{1\} \times [0, 1]) = \rho$  and  $\Delta([0, 1] \times \{0\}) = y$  and  $\Delta|_{[0, 1] \times (0, 1)}$  is an embedding. We abuse the notation by using  $\Delta$  for its image. We can also assume  $\Delta$  is transverse to  $\mathcal{D}$ . Then  $\Delta \cap \mathcal{D}$  would be some collection of circles and some paths  $\{s_j\}$  whose endpoints  $\{q_j^+, q_j^-\}$  are in  $\rho$  as  $\tau \cap \mathcal{D} = \emptyset$ . Each  $p_j^\pm = \psi^{-1}(q_j^\pm) \in D^2$  are in different subdisks  $W_j^\pm$  in  $D^2$  where  $\{W_j^\pm\} \subset \varphi^{-1}(O_i^+)$ . As  $\varphi = \pi \circ \psi$ , there is a one to one correspondence between the points  $p_j^\pm$  which are in  $\varphi^{-1}(y)$  and the disks  $W_j^\pm$ , i.e.  $p_j^\pm \in W_j^\pm$ . Recall that  $\varphi(p_j^\pm) = y$  and  $\varphi(W_j^\pm) = O_i^+$ .

Now, we will modify  $\widehat{\mathcal{D}}$  by pushing it off from  $O_i^+$  and get a new disk  $\widehat{\mathcal{D}}'$  in  $\Omega_{c_n}$  with less area. For a fixed  $j_0$ , let  $B_{j_0}$  be a subdisk in the interior of  $\mathcal{D}$  which contains  $\psi(W_{j_0}^+) \cup \psi(W_{j_0}^-)$ , and no other  $\psi(W_j^\pm)$  for  $j \neq j_0$ . Consider the disk  $\widehat{B}_{j_0} = \pi(B_{j_0})$  in  $S_{c_n}$ . By construction,  $O_i^+ = \pi(W_{j_0}^\pm)$  is in  $\widehat{B}_{j_0}$ . Now,  $\partial \widehat{B}_{j_0}$  consists of two parts, say  $\partial \widehat{B}_{j_0} = \omega_1 \cup \omega_2$  where  $\omega_1 = \pi(\partial B_{j_0})$ , and  $\omega_2 = \partial \widehat{B}_{j_0} - \omega_1$ . By construction,  $\omega_2 \neq \emptyset$ . Notice that  $\omega_2$  consists of points where the geodesic ray starting from those points orthogonal to  $S_{c_n}$  are tangent to  $\mathcal{D}$ . Then by using a version of Sard's theorem, we can find a path  $\gamma$  connecting  $\omega_2$  and  $\partial O_i^+$  such that  $\pi^{-1}(x)$  has even number of preimages (assume exactly 2 for simplicity) with  $\pi^{-1}(x) = \{z_x^+, z_x^-\}$  for any point  $x \in \gamma$  except for a finitely many points  $\{a_1, a_2, \dots, a_k\}$ . Hence,  $\varphi^{-1}(\partial O_i^+ \cup \gamma)$  contains a circle  $\beta$  which bounds a (singular) disk  $F$  in  $D^2$  such that  $\psi(F) \supseteq W_{j_0}^\pm$ . Notice that  $\beta = \beta^+ \cup \beta^- \cup \{\varphi^{-1}(\{a_1, a_2, \dots, a_k\})\}$  where  $\beta^+$  is the arc segment consists of positive preimages  $\{z_x^+\}$ , and  $\beta^-$  is defined similarly.

Now, consider the disk defined as  $\widehat{\mathcal{D}}' = \varphi(D^2 - \text{int}(F)) = \widehat{\mathcal{D}} - \text{int}(O_i^+)$ . Note also that  $\varphi(D^2 - F) = \widehat{\mathcal{D}} - (O_i^+ \cup \gamma)$ . Since  $\varphi(z_x^+) = \varphi(z_x^-)$ , we get a continuous map  $\varphi' : D^2 \rightarrow \Omega_{c_n}$  such that  $\widehat{\mathcal{D}}' = \varphi'(D^2)$ . This is because when you remove a subdisk  $F$  from  $D^2$ , and identify two connected subarcs  $\beta^+$  and  $\beta^-$  in  $\beta = \partial F = \beta^+ \cup \beta^-$ , then the new topological object would be a disk again.

Since the collection of points  $\{a_1, a_2, \dots, a_k\}$  has only one preimage, when we do this surgery in the domain disk, it becomes a topological space  $\Delta$  which is a disk with  $k$  pair of points  $\{b_1^+, b_1^-, \dots, b_k^+, b_k^-\}$  identified. However,  $\Delta$  can be seen as the image of another map  $F : D^2 \rightarrow \Delta$  with  $F(b_i^\pm) = a_i$ . Hence, the surgered object  $\widehat{\mathcal{D}}'$  can be seen as an image of a disk with  $\partial\widehat{\mathcal{D}}' = \alpha_n$  and  $\widehat{\mathcal{D}}' \cap \mathcal{T}_i = \emptyset$ . By construction, there are only finitely many  $i > 0$  with  $\widehat{\Omega}_{c_n} \cap \mathcal{T}_i \neq \emptyset$ , i.e. there exists  $K_n > 0$  such that for any  $i > K_n$ ,  $\widehat{\Omega}_{c_n} \cap \mathcal{T}_i = \emptyset$ . Hence, if we do this surgery for any  $i \leq K_n$ , we get a disk  $\widehat{\mathcal{D}}'$  in  $\Omega_{c_n}$  with  $\partial\widehat{\mathcal{D}}' = \alpha_n$  with  $|\widehat{\mathcal{D}}'| < |E_n|$ . However, this is a contradiction since  $E_n$  is the least area disk in  $\Omega_{c_n}$  with boundary  $\alpha_n$ .  $\square$

## 2.2. The Limit.

In the previous section, we constructed a sequence of least area disks  $\{E_n\}$  in  $X$  with  $\partial E_n = \alpha_n \rightarrow \Gamma$  where  $\Gamma$  is the non-rectifiable curve in  $\partial_\infty X \subset S_\infty^2(\mathbf{H}^3)$  constructed in the previous part. In this section, we show that the sequence of least area disks  $\{E_n\}$  has a subsequence limiting on an embedded least area plane  $\Sigma$  in  $X$  with  $\partial_\infty \Sigma = \Gamma$  by using the techniques of Gabai [Ga]. Since  $\Sigma$  is an embedded least area plane in  $X$ , it will be an embedded minimal plane in  $\mathbf{H}^3$ . In the next section, we will show that  $\Sigma$  is also non-properly embedded in  $\mathbf{H}^3$ , and prove the main result of the paper.

First, we need a definition which we use in the following part. For details of the results and notions in this section, see Section 3 in [Ga].

**Definition 2.2.** The sequence  $\{D_i\}$  of smooth embedded disks in a Riemannian manifold  $X$  converges to the lamination  $\sigma$  if

- $\sigma = \{x = \lim x_i \mid x_i \in D_i, \{x_i\} \text{ is a convergent sequence in } X\}$
- $\sigma = \{x = \lim x_{n_i} \mid x_i \in D_i, \{x_i\} \text{ has a convergent subsequence } \{x_{n_i}\} \text{ in } X\}$
- For any  $x \in \sigma$ , there exists a sequence  $\{x_i\}$  with  $x_i \in D_i$  and  $\lim x_i = x$  such that there exist embeddings  $f_i : D^2 \rightarrow D_i$  which converge in the  $C^\infty$ -topology to a smooth embedding  $f : D^2 \rightarrow L_x$ , where  $x_i \in f_i(\text{Int}(D^2))$ , and  $L_x$  is the leaf of  $\sigma$  through  $x$ , and  $x \in f(\text{Int}(D^2))$ .

We call such a lamination  $\sigma$  a  $D^2$ -limit lamination [Ga].

In other words,  $\{D_i\}$  is a sequence of smooth embedded disks such that the set of the limits of all  $\{x_i\}$  with  $x_i \in D_i$  and the set of the limits of the subsequences are the same. This is a very strong and essential condition on  $\{D_i\}$  in order to limit on a collection of *pairwise disjoint* embedded surfaces. Otherwise, one might simply take a sequence such that  $D_{2i+1} = \Sigma_1$  and  $D_{2i} = \Sigma_2$  where  $\Sigma_1$  and  $\Sigma_2$  are intersecting disks. Then, without the first condition ( $\sigma$  being just the union of limit points),  $\sigma = \Sigma_1 \cup \Sigma_2$  in this case, which is not a collection of pairwise disjoint embedded surfaces. However, the first condition forces  $\sigma$  to be either  $\Sigma_1$  or  $\Sigma_2$ , not the union of them. By similar reasons, this condition is also important to make sure the embeddedness of the disks in the collection  $\sigma$ .

Now, we state the following lemma ([Ga], Lemma 3.3) which is essential for the following part.

**Lemma 2.3.** *If  $\{E_n\}$  is a sequence of embedded least area disks in  $\mathbf{H}^3$ , where  $\partial E_n \rightarrow \infty$ , then after passing to a subsequence  $\{E_{n_j}\}$  converges to a (possibly empty)  $D^2$ -limit lamination  $\sigma$  by least area planes in  $\mathbf{H}^3$ .*

By using this lemma, we will get a least area plane in  $X$  which is constructed as a limit of the sequence of least area disks  $\{E_n\}$  in  $X$ .

**Lemma 2.4.** *The sequence of least area disks  $\{E_n\}$  in  $X$  constructed in the previous section has a subsequence  $\{E_{n_j}\}$  converges to a nonempty  $D^2$ -limit lamination  $\sigma$  by least area planes in  $X$ .*

*Proof:* Even though Lemma 2.3 is stated for  $\mathbf{H}^3$ , since its proof is a local construction, it also applies to our case where the ambient manifold is  $X \subset \mathbf{H}^3$ . Hence, all we need to show is that the lamination we get in the limit is nonempty.

To show that, we construct a sequence of points  $\{x_n\}$  with  $x_n \in E_n$  such that  $\{x_n\}$  has a convergent subsequence  $\{x_{n_j}\}$  with  $x_{n_j} \rightarrow p$  for  $p \in X$ . In the upper half space model for  $\mathbf{H}^3$ , let  $\beta$  be the geodesic segment in  $\mathbf{H}^3$  starting from the point  $(0, 0, \frac{1}{2})$  and ending at the point  $(0, 0, 3)$  (See Figure 3). Let  $\gamma_1$  be the round circle in  $\mathbf{R}^2 \times \{0\} \subset S_\infty^2(\mathbf{H}^3)$  with center  $(0, 0, 0)$  and radius  $\frac{1}{2}$ , and  $\gamma_2$  be the round circle with center  $(0, 0, 0)$  radius 3. Let  $\mathcal{A}$  be the annulus in  $\mathbf{R}^2 \times \{0\}$  bounded by  $\gamma_1$  and  $\gamma_2$ . Then for any  $n$ ,  $\Gamma_n$  would be in  $\mathcal{A}$ . If  $P_i$  is the geodesic plane with  $\partial_\infty P_i = \gamma_i$ , then let  $\mathbf{H}^3 - P_i = \Omega_i^+ \cup \Omega_i^-$  where  $(0, 0, 0) \in \Omega_i^-$ . Let

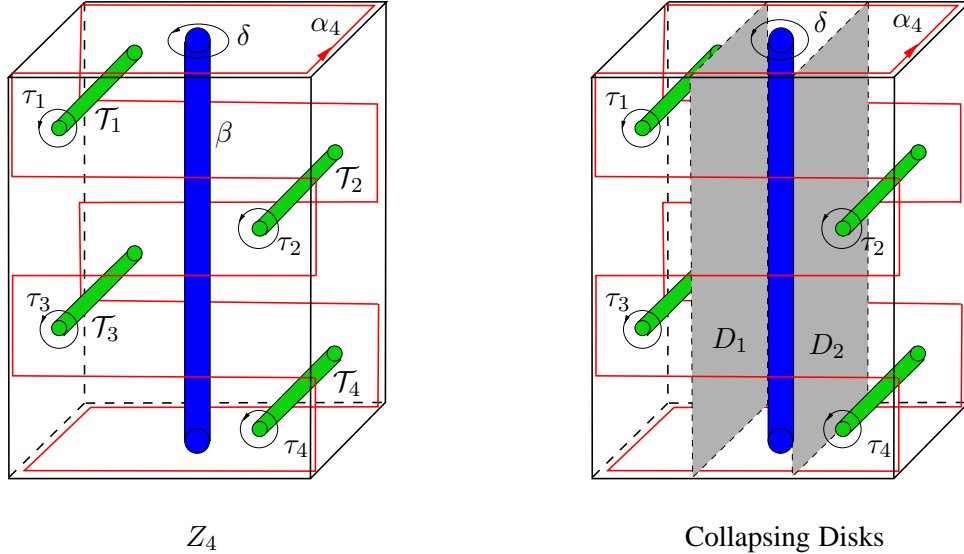


FIGURE 2. The box corresponds to  $Z_4$ . Green tubes are the tunnels, and blue tube is the curve  $\beta$ . The red curve is  $\alpha_4$ . In the right, the grey rectangles are the collapsing disks.



$Y = \Omega_1^+ \cap \Omega_2^-$ . Then,  $\partial_\infty Y = \mathcal{A}$ . Since  $Y$  is convex, then by convex hull property [Co2], for any  $n$ ,  $E_n$  would be in  $Y$ .

We claim that  $E_n \cap \beta \neq \emptyset$  for any  $n$ . Then by taking  $x_n \in \beta \cap E_n$ , we can construct the desired sequence, and finish the proof. We claim that  $\partial E_n = \alpha_n$  links  $\beta$  in  $X$ . Recall that for any  $n$ ,  $E_n \subset Y$ , and  $\partial Y = P_1 \cup P_2$ . Also, the endpoints of  $\beta$  belong to  $P_1$  and  $P_2$ , i.e.  $(0, 0, \frac{1}{2}) \in P_1$  and  $(0, 0, 3) \in P_2$ . Let  $Y_n = Y - \bigcup_{i=1}^n \mathcal{T}_i$ . Clearly,  $Y$  is topologically a 3-ball, and  $Y_n$  is a genus  $n$  handlebody (a 3-ball with  $n$  1-handles attached). For any  $n$ ,  $\alpha_n$  is a trivial loop in  $\pi_1(Y_n)$  by construction. If we realize  $Y_n$  topologically as a 3-ball with  $n$  1-handles attached, and each 1-handle corresponds to a tunnel  $\mathcal{T}_i$ , then  $\pi_1(Y_n)$  would be free product of  $n$  copies of  $\mathbf{Z}$  i.e.  $\pi_1(Y_n) = *_{i=1}^n \mathbf{Z}$ . Hence,  $\pi_1(Y_n) = \langle \tau_1, \tau_2, \dots, \tau_n \rangle$  where  $\tau_i$  is the loop which corresponds to an essential simple closed curve in the annulus  $\partial \mathcal{T}_i$ . Again, by construction,  $\alpha_n$  is a trivial loop in  $Y_n$ .

Let  $Z_n = Y_n - \beta$  be the topologically genus  $n+1$  handlebody which is a 3-ball with  $n+1$  1-handles (See Figure 2). Then,  $\pi_1(Z_n) = \langle \delta, \tau_1, \tau_2, \dots, \tau_n \rangle$  where  $\delta$  is the generator coming from  $\beta$ . i.e.  $\delta$  corresponds to the essential loop of the annulus  $\partial N_\epsilon(\beta) \cap \text{int}(Y)$ . Even though,  $\alpha_n$  is trivial in  $Y_n$ , it is not trivial in  $Z_n$  as  $\alpha_n = \delta \cdot \tau_1 \cdot \delta^{-1} \cdot \tau_1^{-1} \cdot \tau_2 \cdot \delta^{-1} \cdot \tau_2^{-1} \dots \tau_n \cdot \delta^{-1} \cdot \tau_n^{-1}$  which is not a trivial element in  $\pi_1(Z_n)$ . To see this, one might collapse the disks as in Figure 2-right, and divide  $\alpha_n$  to simpler components to write it down explicitly in terms of the generators of  $\pi_1(Z_n)$ . Hence,  $\alpha_n$  is trivial loop in  $Y_n$ , but it is not trivial in  $Z_n$ . This implies that any disk bounding  $\alpha_n$  in  $Y_n$  must intersect  $\beta$ . Hence,  $\beta \cap E_n \neq \emptyset$  for any  $n$ .

Let  $x_n$  be a point in  $\beta \cap E_n$  for any  $n$ . The sequence  $\{x_n\}$  is a subset of compact geodesic segment  $\beta$ . Hence, there is a subsequence  $\{x_{n_j}\}$  with  $x_{n_j} \rightarrow p$  where  $p$  is a point in  $X$ . Now, replace the sequence  $\{E_n\}$  with the subsequence  $\{E_{n_j}\}$ . Then by applying Lemma 2.3 to the new sequence  $\{E_n\}$ , we get a subsequence  $\{E_{n_k}\}$  which limits on a nonempty  $D^2$ -limit lamination  $\sigma$  by least area planes in  $X$ . The proof follows.  $\square$

By the proof of Lemma 2.3, for any leaf  $L$  in  $\sigma$ , for any subdisk  $D$  in  $L$ , we can find sufficiently close disk  $D_n$  in the disk  $D_n$ . Since  $\partial E_n = \alpha_n \rightarrow \Gamma$ , we can find a least area plane  $\Sigma$  in  $\sigma$  with  $\partial_\infty \Sigma = \Gamma$ . Note that again by construction  $\partial_\infty \sigma = \overline{\Gamma}$  where  $\overline{\Gamma}$  is the closure of  $\Gamma$  in  $S_\infty^2(\mathbf{H}^3)$ . By construction of  $\Gamma$ ,  $\overline{\Gamma}$  would be  $\Gamma \cup \gamma$  where  $\gamma$  is the unit circle in  $\mathbf{R}^2 \times \{0\} \subset S_\infty^2(\mathbf{H}^3)$  with center  $(0, 0, 0)$ . Also, by varying the transverse geodesic segment  $\beta$ , it is not hard to show that the geodesic plane  $P$  with  $\partial_\infty P = \gamma$  is another leaf of the lamination  $\sigma$ .

### 2.3. Non-properly Embeddedness.

In this section, we will show that the least area plane  $\Sigma$  in  $X$  constructed in the previous section is not properly embedded (See Figure 3). Hence, this will show that  $\Sigma$  is a non-properly embedded minimal plane in  $\mathbf{H}^3$ , and the main result of the paper follows.

**Theorem 2.5.** *There exists a non-properly embedded, complete minimal plane in  $\mathbf{H}^3$ .*

*Proof:* We claim that the least area plane  $\Sigma$  in  $X$  constructed in previous section is not properly embedded. Since  $\Sigma$  is a least area plane in  $X$ , it is automatically a minimal plane in  $\mathbf{H}^3$ . Hence, if we show that  $\Sigma$  is not properly embedded in  $\mathbf{H}^3$ , we are done.

Assume that  $\Sigma$  is properly embedded. Let  $\beta$  be as in the proof of Lemma 2.4 (See Figure 3), i.e. In the upper half space model for  $\mathbf{H}^3$ ,  $\beta$  is the geodesic segment in  $\mathbf{H}^3$  starting from the point  $(0, 0, \frac{1}{2})$  and ending at the point  $(0, 0, 3)$ . If  $\beta$  is not transverse to  $\Sigma$ , modify  $\beta$  slightly at non-transverse points to make it transverse to  $\Sigma$ . Now, as  $\Sigma$  is properly embedded,  $\Sigma \cap \beta$  is compact. Let  $\gamma$  be the unit circle in  $\mathbf{R}^2 \times \{0\} \subset S_\infty^2(\mathbf{H}^3)$  with center  $(0, 0, 0)$ , and  $P$  be the geodesic plane in  $\mathbf{H}^3$  with  $\partial_\infty P = \gamma$ . As mentioned in the previous section  $P$  is also a least area plane in the lamination  $\sigma$ , and hence  $\Sigma \cap P = \emptyset$ . Since  $\Sigma \cap \beta$  is compact, this implies  $\delta = \inf_{p \in \Sigma \cap \beta} \{p_z \in \mathbf{R} \mid p = (p_x, p_y, p_z)\} > 1$ .

Now, we claim that there is another point  $q \in \beta \cap \Sigma$  with  $1 < q_z < \delta$  which gives us a contradiction. Let  $N > 0$  be such that  $1 + \frac{1}{N} < \delta$ . Recall that  $\Gamma_n = \Gamma_{n-1} \# C_n$  where  $C_n$  is the round circle in  $\mathbf{R}^2 \times \{0\} \subset S_\infty^2(\mathbf{H}^3)$  with radius  $1 + \frac{1}{n}$  and center  $(0, 0, 0)$ . Let  $\mu_1$  be the round circle in  $\mathbf{R}^2 \times \{0\} \subset S_\infty^2(\mathbf{H}^3)$  with radius  $\frac{1}{N+1} < r_1 < \frac{1}{N}$  and center  $(0, 0, 0)$ . Let  $\mu_2$  be the round circle in  $\mathbf{R}^2 \times \{0\} \subset S_\infty^2(\mathbf{H}^3)$  with radius  $\frac{1}{2N+1} < r_2 < \frac{1}{2N}$  and center  $(0, 0, 0)$ . Hence,  $\mu_1$  is between the circles  $C_N$  and  $C_{N+1}$  in  $\mathbf{R}^2 \times \{0\}$ , and  $\mu_2$  is between the circles  $C_{2N}$  and  $C_{2N+1}$  in  $\mathbf{R}^2 \times \{0\}$ . By choosing  $r_1, r_2$  accordingly, further assume that  $\mu_1 \cap \eta_N^\pm = \emptyset$  and  $\mu_2 \cap \eta_{2N}^\pm = \emptyset$  (See Figure 1). Let  $P_i$  be the geodesic plane in  $\mathbf{H}^3$  with  $\partial_\infty P_i = \mu_i$ , and let  $\mathbf{H}^3 - P_i = \Omega_i^+ \cup \Omega_i^-$  where  $(0, 0, 0) \in \Omega_i^-$ . Then,  $P_1$  and  $P_2$  are least area planes in  $X$ , too.

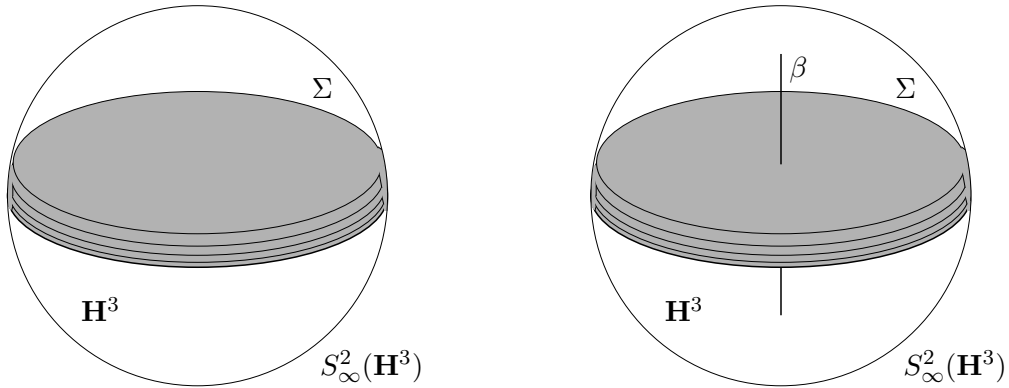


FIGURE 3. In the figure left, we see the minimal plane  $\Sigma$  with  $\partial_\infty \Sigma = \Gamma$ . In the figure right, we see the line segment  $\beta$  which is transverse to  $\Sigma$ .

Now, consider  $\Sigma \cap P_i$ . First,  $\partial_\infty \Sigma = \Gamma$  and  $\partial_\infty P_i = \mu_i$ .  $\mu_i$  intersect  $\Gamma$  at exactly 2 points by construction. Say  $\mu_i \cap \Gamma = \{x_i^+, x_i^-\}$ . Since  $\Sigma$  and  $P_i$  are least area planes,  $\Sigma \cap P_i$  cannot contain a simple closed curve by Meeks-Yau exchange roundoff trick [Co2]. Hence,  $\Sigma \cap P_i = \{l_j^i\}$  where  $l_j^i$  is an infinite line segment in  $\mathbf{H}^3$  with  $\partial_\infty l_j^i = \{x_i^+, x_i^-\}$ . Since  $P_i$  is separating in  $\mathbf{H}^3$ , all lines are separating in  $\Sigma$ . Hence, there is a natural ordering among  $\{l_j^i\}$ . Let  $l_1$  be the lowermost line among  $\{l_j^1\}$  and let  $l_2$  be the uppermost line among  $\{l_j^2\}$  such that  $l_1 \cup l_2$  separates a component  $\widehat{\Sigma}$  in  $\Sigma$  where  $\widehat{\Sigma}$  contains no line segments  $\{l_j^i\}$ .

Now,  $P_1$  and  $P_2$  are the geodesic planes with  $\partial_\infty P_i = \mu_i$ , then let  $\mathbf{H}^3 - P_i = \Omega_i^+ \cup \Omega_i^-$  where  $(0, 0, 0) \in \Omega_i^-$ . Let  $Y = \overline{\Omega_1^+} \cap \overline{\Omega_2^-}$ . By assumption on  $r_1$  and  $r_2$ , we know that  $Y \cap \mathcal{T}_N = \emptyset$  and  $Y \cap \mathcal{T}_{2N} = \emptyset$ . Hence,  $\widehat{Y} = Y \cap X$  would be a genus  $N - 1$  handlebody, i.e.  $\widehat{Y} = Y - \bigcup_{i=N+1}^{2N-1} \mathcal{T}_i$ . By construction,  $\widehat{\Sigma} \subset \widehat{Y}$ . Topologically, we have a closed disk  $\widehat{\Sigma}$  with  $\partial \widehat{\Sigma} = \widehat{\alpha} \subset \partial \widehat{Y}$ . Hence,  $\widehat{\alpha}$  is trivial element in  $\pi_1(\widehat{Y})$ . However, if we define  $\widehat{Z} = \widehat{Y} - \beta$ , as in the proof of Lemma 2.4, we see that  $\widehat{\alpha}$  is not a trivial element in  $\pi_1(\widehat{Z})$ . This proves that  $\widehat{\Sigma} \cap \beta \neq \emptyset$ . Let  $q$  be a point in  $\widehat{\Sigma} \cap \beta$ . By construction  $\frac{1}{2N} < q_z < \frac{1}{N} < \delta$ . However, this contradicts with the definition of  $\delta$  as  $\delta = \inf_{p \in \Sigma \cap \beta} \{p_z \in \mathbf{R} \mid p = (p_x, p_y, p_z)\}$ . The proof follows.  $\square$

### 3. FINAL REMARKS

We should note that our construction differs from the Freedman and He's heuristic construction in the following way. In their construction, they want to apply to bridge principle to construct the sequence of minimal disks, then take the limit. However, the examples of curves in  $S_\infty^2(\mathbf{H}^3)$  described in [La] shows that such a limit might not give a connected plane, and bridges might escape to infinity. In our construction, it can be thought that we are still using the bridges, but we are also using the tunnels acting as barrier which prevents bridges to escape to infinity. However, because of these tunnels, while  $\Sigma$  is a least area plane in  $X$  ( $\mathbf{H}^3$  with tunnels deleted), it is not a least area plane in  $\mathbf{H}^3$  anymore by [Co1].

On the other hand, one might try to use a "bridge principle at infinity" to construct such an example. In other words, one might start with infinite family of geodesic planes as in this paper, and try to build bridges in  $S_\infty^2(\mathbf{H}^3)$  to connect the asymptotic boundaries of the geodesic planes, and take the limit. However, the problem with this approach would be when you make a bridge at infinity, one might completely lost the original geodesic planes which goes through the compact part, and get a completely different least area plane with the new asymptotic boundary which stays close to the asymptotic sphere  $S_\infty^2(\mathbf{H}^3)$ . Hence, the barrier tunnels in our construction are very essential to construct such an example.

Note also that it is known that if  $\Sigma$  is a least area plane in  $\mathbf{H}^3$  with  $\partial_\infty \Sigma = \Gamma$  where  $\Gamma$  is a simple closed curve which contains at least one smooth point, then  $\Sigma$  is properly embedded in  $\mathbf{H}^3$  by [Co1]. However, since neither  $\Sigma$  is not least area in our example nor  $\Gamma$  is a simple closed curve, [Co1] does not apply here. Also, even though our example gives a complete, non-properly embedded minimal plane

in  $\mathbf{H}^3$ , it is still not known the existence of a complete, non-properly embedded least area plane in  $\mathbf{H}^3$ . So, it would be an interesting question whether there exists a non-properly embedded least area plane in  $\mathbf{H}^3$ .

As it is stated in the introduction, the key lemma of [MR1] to prove its main result that *a complete, embedded minimal surfaces with finite genus in  $\mathbf{R}^3$  is proper* is also true for 3-manifolds with non-positive curvature. However, the example constructed in this paper shows that even though the key lemma (Theorem 1 in [MR1]) is valid for minimal surfaces in  $\mathbf{H}^3$ , it does not imply the properly embeddedness in  $\mathbf{H}^3$  like in  $\mathbf{R}^3$ .

If one considers our example constructed in this paper in the [MR1] context, we get the following picture.  $\Sigma$  (in Theorem 2.5) is a minimal plane in  $\mathbf{H}^3$  with positive injectivity radius. If one applies Theorem 1 (or Theorem 3) of [MR1] to  $\Sigma$  in  $\mathbf{H}^3$ , we get a lamination  $\sigma = \overline{\Sigma} = \Sigma \cup P$  where  $P$  is the geodesic plane whose asymptotic boundary is the unit circle in  $\mathbf{R}^2 \times \{0\} \subset S_\infty^2(\mathbf{H}^3)$ . By using Theorem 1, they prove Theorem 2 in [MR1] which states that *a complete embedded connected minimal surface in  $\mathbf{R}^3$  with positive injectivity radius is always properly embedded*.

Our example  $\Sigma$  shows that Theorem 2 of [MR1] is not true in  $\mathbf{H}^3$ , whereas Theorem 1 of [MR1] is valid in  $\mathbf{H}^3$ . Now, what goes wrong to get properly embeddedness of  $\Sigma$  in  $\mathbf{H}^3$  as in the case of  $\mathbf{R}^3$  in [MR1]? In the proof of Theorem 2 in [MR1], Meeks and Rosenberg apply Theorem 1 to a complete embedded minimal surface with positive injectivity radius in  $\mathbf{R}^3$ , and in the closure, they get a minimal lamination  $\mathcal{L}$ . By [MR2], they concluded that the limit leaves must be planes in  $\mathbf{R}^3$ . Similarly, the limit leaf  $P$  in our lamination  $\sigma$  is also a plane in  $\mathbf{H}^3$ . So, everything is similar so far. However, when you apply Theorem 4 to  $\mathcal{L}$ , they show that  $M$  must have bounded curvature in an  $\epsilon$  neighborhood of the limit leaf. However, this contradicts to Lemma 1.3 of [MR2] which states that  $M$  cannot have unbounded curvature in the neighborhood of a limit leaf. On the other hand, Theorem 4 is valid for  $\sigma$  in  $\mathbf{H}^3$ , too. Hence, we get that  $\Sigma$  must have bounded curvature in  $\epsilon$  neighborhood of the limit leaf  $P$ . Unlike  $\mathbf{R}^3$ , this can happen in  $\mathbf{H}^3$  case as the high curvature regions which corresponds to *the bridges* in  $\Sigma$  are far away from the limit leaf in  $\mathbf{H}^3$ . So, an analogous result of Lemma 1.3 in [MR2] is not true in  $\mathbf{H}^3$  in general, and this is the place where the technique in [MR1] breaks down in  $\mathbf{H}^3$  case.

Note also that in [MT], Meeks and Tinaglia recently announced examples of non-properly embedded constant mean curvature surfaces of finite topology for any  $H \in [0, 1)$  in  $\mathbf{H}^3$ . They also show that if  $H \geq 1$  then the surface must be properly embedded in  $\mathbf{H}^3$ . Their example is different than ours, as they construct an infinite strip which is a constant mean curvature surface limiting into two constant mean curvature annuli in  $\mathbf{H}^3$ . The asymptotic boundary of this surface is a pair of infinite lines where each line spirals into a pair of circles (asymptotic boundaries of the annuli).

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